

Exact Solution of Couple Burgers' Equation using Cubic B-Spline Collocation Method for Fluid Suspension/Colloid under the Influence of Gravity

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Abstract

In this research, Cubic B-Spline plane method was used to solve numerically the one-dimensional Burger's Equation with initial condition ; $u(x, 0) = \varphi_1(x), v(x, 0) = \varphi_2(x)$, boundary conditions; $u(a, t) = f_1(a, t), u(b, t) = f_2(b, t) \quad v(a, t) = g_1(a, t), v(b, t) = g_2(b, t)$. The cubic trigonometric B-spline was used for interpolating the solutions at each time and using the Von-Neumann method to check the stability. The obtained numerical result showed that the method was efficient, robust and reliable for solving Burgers' Equation accurately even involving high Reynolds numbers for which the exact solutions have failed.

Keywords: Cubic B-Spline, Burger's Equation, Fluid Suspension, Gravity.

Introduction

Numerical solution of Burgers' equation has been used with different numerical techniques in the past because of its wide range of applicability in different fields of science, technology and engineering research (Kutluay, *et al.*, 2007; Abdou and Soliman, 2005). A hybrid numerical order centered on Euler implicit method, quasilinearization and uniform Haar wavelets were employed by (Jiwari, 2015).

The coupled viscous Burgers' equation was derived by Esipov, (1995) and is a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids under the effect of gravity (Esipov, 1995). Recently, numerical solutions to coupled Burgers' equation have attracted a lot of attention from researchers globally and has given rise to in a number of numerical algorithms. For instance, cubic B-spline collocation scheme (Mittal and Arora, 2011), Chebyshev spectral collocation method (Khater *et al.*, 2008), fully implicit finite difference scheme (Sirvastava *et al.*, 2013; Ackerman and Phillips, 1972) and many more have been developed for the numerical solution of coupled Burgers' equations.

Even if numerical simulation of the Burgers' Equation is well documented in the literature, gaps still exist. For comparative discussion regarding the physical and mathematical significance of the Burgers' Equation as can be seen in situation where fluid equations have

global solution, however, one may still ask about their asymptotic behavior at larger time. This is because the solutions converge as time goes to infinity in a function space.

A system of two-dimensional Burgers' equations is represented in two different formulae:

$$\frac{\partial v}{\partial t} + \nabla \frac{\partial(v, v)}{2} = \varepsilon \Delta v \quad (1)$$

or

$$\frac{\partial v}{\partial t} + (v, \nabla v) = \varepsilon \Delta v \quad (2)$$

Where $v = (u(x, y, t), v(x, y, t)) \in R^2$. In general, $\Delta(v, v)/2$ and $(v, \Delta v)$ are different; however, for the potential case, they are equivalent (i.e. there exists a scalar function H , such that $v = \nabla H$). These two-dimensional Burgers' equations have been extensively used to model procedures in soil hydrography such as the consolidation of wet soil.

Particularly, in this paper we employed the following system of two-dimensional Burgers' equations:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \varepsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (x, y, t) \in D \times (0, T] \quad (3)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \varepsilon \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \quad (x, y, t) \in D \times (0, T] \quad (4)$$

Subject to the initial conditions

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in D \quad (5)$$

$$v(x, y, 0) = v_0(x, y), \quad (x, y) \in D \quad (6)$$

The boundary conditions

$$u(x, y, t) = f(x, y, t), \quad (x, y, t) \in \partial D \times (0, T] \quad (7)$$

$$v(x, y, t) = f(x, y, t), \quad (x, y, t) \in \partial D \times (0, T] \quad (8)$$

And the potential symmetry condition

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \quad (9)$$

where $D = [a, b] \times [c, d]$ is a rectangular domain and ∂D is its boundary.

Any implicit time-integration scheme often results in a nonlinear algebraic system, which is usually solved by iterative method or predator-corrector method, for example, the nonlinear system is solved by Newton's method (Wu, *et al.* 2008), and the nonlinear system is solved by both Newton's method and predictor-corrector methods. To overcome this difficulty, we employed numerical solution of coupled viscous Burgers' equation by Esipov, 1995, to study the model of poly-dispersive sedimentation. The coupled viscous Burgers' equations are given by:

$$u_t - u_{xx} + \eta uu_x + \alpha(uv)_x = 0 \quad x \in [a, b], t \in [0, T] \quad (10)$$

$$v_t - v_{xx} + \eta vv_x + \beta(uv)_x = 0 \quad x \in [a, b], t \in [0, T] \quad (11)$$

With initial condition

$$u(x, 0) = \varphi_1(x), \quad v(x, 0) = \varphi_2(x)$$

And boundary conditions

$$u(a, t) = f_1(a, t), \quad u(b, t) = f_2(b, t)$$

$$v(a, t) = g_1(a, t), \quad v(b, t) = g_2(b, t)$$

Where η a real constant is while α and β are arbitrary constants depending on the parameters.

Methodology

This is a simple model of sedimentation of two kinds of particles in fluid suspension or colloids under the influence of gravity. The coupled viscous Burgers' equation is given by:

$$u_t - u_{xx} + \eta uu_x + \alpha(uv)_x = 0 \quad x \in [a, b], t \in [0, T] \quad (12a)$$

$$v_t - v_{xx} + \eta vv_x + \beta(uv)_x = 0 \quad x \in [a, b], t \in [0, T] \quad (13a)$$

With initial condition

$$u(x, 0) = \varphi_1(x), \quad v(x, 0) = \varphi_2(x)$$

and boundary conditions

$$u(a, t) = f_1(a, t), \quad u(b, t) = f_2(b, t)$$

$$v(a, t) = g_1(a, t), \quad v(b, t) = g_2(b, t)$$

Where η is a real constant while α and β are arbitrary constants depending on the parameters.

The exact solution of coupled Burgers' equation was obtained by Kaya using Adomian decomposition method and solution, presented modified extended tanh-function method to obtain exact solution (Mittal, *et al.* 2015; Kadalbajoo *et al.*, 2005)

This paper seeks to find numerical solution for coupled Burgers' equation using cubic B-spline collocation method on space integration and finite difference Crank-Nicolson method on the time integration. The paper is organized as follows: introduction of the general Burgers' equation model and its applications; highlighting the basis function for cubic B-spline and their derivatives at the nodal point, computing approximate solutions in terms of cubic spline function and sometime dependent parameter. The stability of our scheme was tested using Von Neumann method.

Cubic Spline Bases

Constructing the numerical solution, we consider the nodal points (x_i, t_j) defined on the region $[a, b] \times [0, T]$ where:

$$a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$$

$$x_{i+1} - x_i = h, \quad 0 < t_0 < t_1 < \dots < t$$

$$t_{i+1} - t_i = \Delta t$$

The following signifies the cubic B-Spline basis functions at their nodal points:

$$B_m(x) = \frac{1}{h^3} \begin{cases} (x - x_{m-2})^3 & x \in [x_{m-2}, x_{m-1}) \\ (x - x_{m-2})^3 - 4(x - x_{m-1})^3 & x \in [x_{m-1}, x_m) \\ (x_{m+2} - x)^3 - 4(x_{m+1} - x)^3 & x \in [x_m, x_{m+1}) \\ (x_{m+2} - x)^3 & x \in [x_{m+1}, x_{m+2}) \\ 0 & \text{otherwise} \end{cases} \quad (12b)$$

Using this B-Spline basis function, the values of $B_m(x)$ and their derivatives could be obtained from nodal points thus:

At x_{m-2} nodal point we have

$$\frac{1}{h^3} [(x_{m-2} - x_{m-2})^3] = 0 = B_m(x_{m-2})$$

At x_{m-1} nodal point we have

$$B_m(x_{m-1}) = \frac{1}{h^3} [(x_{m-1} - x_{m-2})^3 - 4(x_{m-1} - x_{m-1})^3] = \frac{1}{h^3} \times h^3 - 4(0) = 1$$

At x_m nodal point we have

$$\begin{aligned} B_m(x_m) &= \frac{1}{h^3} [(x_{m+2} - x_m)^3 - 4(x_{m+1} - x_m)^3] \\ &= \frac{1}{h^3} [(x_{m+2} - x_{m+1}) - (x_{m+1} - x_m)]^3 - 4(x_{m+1} - x_m)^3 \left[\frac{1}{h^3} (2h)^3 - 4h^3 \right] = \frac{1}{h^3} \times 4h^3 \\ &= 4 \end{aligned}$$

At x_{m+1} nodal point we have

$$\begin{aligned} B_m(x_{m+1}) &= \frac{1}{h^3} [(x_{m+2} - x_{m+1})^3] = \frac{1}{h^3} \times h^3 = 1 \\ B_m(x_{m+2}) &= 0 \end{aligned}$$

1st Derivatives of cubic spline at the nodal point:

$$\begin{aligned} B'_m(x_{m-2}) &= \frac{1}{h^3} [3(x_{m-2} - x_{m-2})^2] = 0 \\ B'_m(x_{m-1}) &= \frac{1}{h^3} [3(x_{m-1} - x_{m-2})^2 - 12(x_{m-1} - x_{m-1})^2] \\ &= \frac{1}{h^3} \times 3(h^2) = \frac{3}{h} \\ B'_m(x_m) &= \frac{1}{h^3} [3(x_{m+2} - x_m)^2 - 12(x_{m+1} - x_m)^2] \\ &= \frac{1}{h^3} \times [3(2h)^2 - 12h^2] = \frac{1}{h^3} \times 0 = 0 \\ B'_m(x_{m+1}) &= \frac{1}{h^3} [3(x_{m+2} - x_{m+1})^2] = \frac{1}{h^3} \times 3h^2 = \frac{3}{h} \\ B'_m(x_{m+2}) &= 0 \end{aligned}$$

2nd Derivatives at Nodal Points

$$\begin{aligned} B''_m(x_{m-2}) &= 0 \\ B''_m(x_{m-1}) &= \frac{6}{h^3} [(x_{m-1} - x_{m-2}) - 24(x_{m-1} - x_{m-1})] = \frac{6}{h^2} \\ B''_m(x_m) &= \frac{1}{h^3} [6(2h) - 24h] = \frac{1}{h^3} (-12) = -\frac{12}{h^2} \end{aligned}$$

$$B_m''(x_{m+1}) = \frac{1}{h^3} [6h] = \frac{6}{h^2}$$

$$B_m''(x_{m+2}) = 0.$$

The Solution of Coupled Viscous Burgers' Equation

Here we discretize the time derivative of equation (12a) and (13a) using finite difference scheme of the Crank Nicolson scheme to obtain

$$\frac{u^{n+1} - u^n}{\Delta t} - \frac{[u_{xx}^{n+1} + u_{xx}^n]}{2} + \eta \frac{[(uu_x)^{n+1} + (uu_x)^n]}{2} + \alpha \frac{[((uv)_x)^{n+1} + ((uv)_x)^n]}{2} = 0 \quad (13b)$$

$$\frac{v^{n+1} - v^n}{\Delta t} - \frac{[v_{xx}^{n+1} + v_{xx}^n]}{2} + \eta \frac{[(vv_x)^{n+1} + (vv_x)^n]}{2} + \beta \frac{[((uv)_x)^{n+1} + ((uv)_x)^n]}{2} = 0 \quad (14)$$

The nonlinearities in (13a) and (12b) are linearized thus:

$$(uu_x)^{n+1} = u^{n+1}u_x^n + u^n u_x^{n+1} - (uu_x)^n \quad (15)$$

$(uv)_x^{n+1} + (uv)_x^n$ is computed thus:

$$(uv)_x = vu_x + uv_x$$

So that

$$[(uv_x)^{n+1} + (uv_x)^n] + [(vu_x)^{n+1} + (vu_x)^n] = (uv)_x^{n+1} + (uv)_x^n \quad (16)$$

From (14) we substitute into (16), the like terms cancel thus:

$$\begin{aligned} (uv)_x^{n+1} + (uv)_x^n &= (u^{n+1}v_x^n + u^n v_x^{n+1}) - (uv_x)^n + (uv_x)^n + (v^{n+1}u_x^n) \\ &+ (v^n u_x^{n+1}) - (vu_x)^n + (vu_x)^n \\ &= u^{n+1}v_x^n + u^n v_x^{n+1} + v^{n+1}u_x^n + v^n u_x^{n+1} \end{aligned} \quad (17)$$

Now, the following equation represents a simplified version of (13b) for what follows:

$$\begin{aligned} \frac{u^{n+1} - u^n}{\Delta t} - \frac{[u_{xx}^{n+1} + u_{xx}^n]}{2} + \eta \frac{[u^{n+1}u_x^n + u^n u_x^{n+1}]}{2} \\ + \alpha \frac{[u^{n+1}v_x^n + u^n v_x^{n+1} + v^{n+1}u_x^n + v^n u_x^{n+1}]}{2} \\ = 0 \end{aligned}$$

$$\begin{aligned} \frac{v^{n+1} - v^n}{\Delta t} - \frac{[v_{xx}^{n+1} + v_{xx}^n]}{2} + \eta \frac{[v^{n+1}v_x^n + v^n v_x^{n+1}]}{2} \\ + \beta \frac{[u^{n+1}v_x^n + u^n v_x^{n+1} + v^{n+1}u_x^n + v^n u_x^{n+1}]}{2} = 0 \end{aligned}$$

Using Cubic B-Spline basis function $B_m(x)$ and the time dependent parameters $\delta_m(t)$ and $\sigma_m(t)$ for $u(x, y)$ and $v(x, t)$ we obtain approximate solutions thus:

$$v_m(x, t) = \sum_{m=-1}^{N+1} \sigma_m(t) B_m(x), \quad u_m(x, t) = \sum_{m=-1}^{N+1} \delta_m(t) B_m(x). \quad (18)$$

We use the Cubic B-Spline function $B_m(x)$ to deduce the approximate solution $u(x)$ and $v(x)$ in terms of time parameter of $u(x, t)$ and $v(x, t)$ respectively.

We compute

$$u_m = \sum_{M=m-2}^{m+2} \delta_M(t) B_M(x);$$

by taking into cognizance the property that $B_{m-i}(x) = B(x_{m-1})$ so that

$$\begin{aligned} u_m &= \delta_{m-2}(t)B_m(x_{m-2}) + \delta_{m-1}(t)B_m(x_{m-1}) + \delta_m(t)B_m(x_m) + \delta_{m+1}(t)B_m(x_{m+1}) \\ &\quad + \delta_{m+2}(t)B_m(x_{m+2}) \\ &= \delta_{m-1} + 4\delta_m + \delta_{m+1} \\ u'_m &= \sum_{M=m-2}^{m+2} \delta_M(t)B'_M(x) \\ &= \delta_{m-2}(t)B'_m(x_{m-2}) + \delta_{m-1}(t)B'_m(x_{m-1}) + \delta_m(t)B'_m(x_m) \\ &\quad + \delta_{m+1}(t)B'_m(x_{m+1}) + \delta_{m+2}(t)B'_m(x_{m+2}) \\ &= 0 + \frac{3}{h}\delta_{m-1} + 0 + \frac{3}{h}\delta_{m+1}(t) + 0 \\ &= \frac{3}{h}[\delta_{m-1}(t) + \delta_{m+1}(t)] \\ u''_m &= \sum_{m=m-2}^{m+2} \delta_m(t)B''_m(x) \\ &= \delta_{m-2}(t)B''_m(x_{m-2}) + \delta_{m-1}(t)B''_m(x_{m-1}) + \delta_m(t)B''_m(x_m) \\ &\quad + \delta_{m+1}(t)B''_m(x_{m+1}) + \delta_{m+2}(t)B''_m(x_{m+2}) \\ &= 0 + \frac{6}{h^2}\delta_{m-1}(t) - \frac{12}{h^2}\delta_m(t) + \frac{6}{h^2}\delta_{m+1} = \frac{6}{h^2}(\delta_{m-1} - 2\delta_m + \delta_{m+1}). \end{aligned}$$

Similarly,

$$\begin{aligned} v_m &= \sigma_{m-1}(t) + 4\sigma_m(t) + \sigma_{m+1}(t) \\ v'_m &= \frac{3}{h}(\sigma_{m+1} - \sigma_{m-1}) \\ v''_m &= \frac{6}{h^2}(\sigma_{m-1} - 2\sigma_m + \sigma_{m+1}) \end{aligned}$$

Next, we plug in the approximate solutions and their derivatives into (13) and (14) thus:

$$\begin{aligned} \Rightarrow & \delta_{m-1} + 4\delta_m + \delta_{m+1} - u^n - \frac{\Delta t}{2} \left[\frac{6}{h^2}(\delta_{m-1} - 2\delta_m + \delta_{m+1}) + u''_{xx} \right] \\ & + \frac{\eta \Delta t}{2} \left[(\delta_{m-1} - 4\delta_m + \delta_{m+1})u'_x + \frac{3}{h}(\delta_{m+1} - \delta_{m-1})u^n \right] \\ & + \frac{\alpha \Delta t}{2} \left[(\delta_{m-1} + 4\delta_m + \delta_{m+1})v'_x + \frac{3}{h}(\sigma_{m+1} - \sigma_{m-1})u^n \right. \\ & \quad \left. + (\sigma_{m-1} + 4\sigma_m + \sigma_{m+1})u'_x + \frac{3}{h}(\delta_{m-1} + \delta_{m+1})v^n \right] = 0 \\ u^n + \frac{\Delta t}{2} u''_{xx} &= (\delta_{m-1} + 4\delta_m + \delta_{m+1}) \left(1 + \frac{\eta \Delta t}{2} u'_x + \frac{\alpha \Delta t}{2} v'_x \right) - \frac{3\Delta t}{h^2} (\delta_{m-1} - 2\delta_m + \\ & \delta_{m+1}) + \left(\frac{3\alpha \Delta t}{2h} v^n + \frac{3\eta \Delta t}{2h} u^n \right) (\delta_{m+1} - \delta_{m-1}) + \frac{3\alpha \Delta t}{2h} (\sigma_{m+1} - \sigma_{m-1})u^n + \frac{\alpha \Delta t}{2} u'_x (\sigma_{m-1} + \\ & 4\sigma_m + \sigma_{m+1}). \end{aligned}$$

Let $a_1 = 1 + \frac{\Delta t}{2}(\eta u_x^n + \alpha v_x^n)$, $a_2 = -\frac{3\Delta t}{h^2}$, $a_3 = \frac{3\Delta t}{2h}(\alpha v^n + \eta u^n)$, $a_4 = \frac{3\Delta t}{2h}(\alpha u^n)$, $a_5 = \frac{\Delta t}{2}(\alpha u_x^n)$

Similarly, from (13a) we substitute the approximate solution $v_m = \sum_{M=m-2}^{m+2} \sigma_M(t) B_M'$ of $v(t, x)$ thus:

$$\begin{aligned} & (\sigma_{m-1} + 4\sigma_m + \sigma_{m+1}) - v^n - \frac{\Delta t}{2} \left[\frac{6}{h^2} (\sigma_{m-1} - 2\sigma_m + \sigma_{m+1}) - v_{xx}^n \right] \\ & + \frac{\eta \Delta t}{2} \left[(\sigma_{m-1} - 2\sigma_m + \sigma_{m+1}) v_x^n + \frac{3}{h} (\sigma_{m+1} + \sigma_{m-1}) v^n \right] \\ & + \frac{\beta \Delta t}{2} \left[(\delta_{m-1} + 4\delta_m + \delta_{m+1}) v_x^n + \frac{3}{h} (\sigma_{m+1} + \sigma_{m-1}) u^n \right. \\ & \left. + (\sigma_{m-1} + 4\sigma_m + \sigma_{m+1}) u_x^n + \frac{3}{h} (\delta_{m+1} + \delta_{m-1}) v^n \right] = 0 \\ v^n + \frac{\Delta t}{2} v_{xx}^n & = \left(1 + \frac{\Delta t}{2} (\eta v_x^n + \beta u_x^n) \right) (\sigma_{m-1} + 4\sigma_m + \sigma_{m+1}) \\ & - \frac{3\Delta t}{h^2} (\sigma_{m-1} - 2\sigma_m + \sigma_{m+1}) + \frac{3\Delta t}{2h} (\sigma_{m+1} + \sigma_{m-1}) (\eta v^n + \beta u^n) \\ & + \frac{3\Delta t}{2h} (\beta v^n) (\delta_{m+1} + \delta_{m-1}) + \frac{\Delta t}{2} (\beta v_x^n) (\delta_{m-1} + 4\delta_m + \delta_{m+1}) \end{aligned}$$

Here,

$$\begin{aligned} a_6 & = \left(1 + \frac{\Delta t}{2} (\eta v_x^n + \beta u_x^n) \right) \\ a_7 & = -\frac{3\Delta t}{h^2} \\ a_8 & = \frac{3\Delta t}{2h} (\eta v^n + \beta u^n) \\ a_9 & = \frac{3\Delta t}{2h} (\beta v^n) \\ a_{10} & = \frac{\Delta t}{2} (\beta v_x^n) \end{aligned}$$

We write the difference equations in the following way:

$$\begin{aligned} & a_1(\delta_{m-1} + 4\delta_m + \delta_{m+1}) + a_2(\delta_{m-1} - 2\delta_m + \delta_{m+1}) + a_3(\delta_{m+1} + \delta_{m-1}) \\ & + a_4(\sigma_{m+1} + \sigma_{m-1}) + a_5(\sigma_{m-1} + 4\sigma_m + \sigma_{m+1}) \\ & = u^n + \frac{\Delta t}{2} u_{xx}^n \end{aligned} \quad (19)$$

$$\begin{aligned} & a_6(\sigma_{m-1} + 4\sigma_m + \sigma_{m+1}) + a_7(\sigma_{m-1} - 2\sigma_m + \sigma_{m+1}) + a_8(\sigma_{m+1} + \sigma_{m-1}) \\ & + a_9(\delta_{m+1} + \delta_{m-1}) + a_{10}(\sigma_{m-1} + 4\sigma_m + \sigma_{m+1}) \\ & = v^n + \frac{\Delta t}{2} v_{xx}^n \end{aligned} \quad (20)$$

$m = 0, \dots, N$

It is noteworthy that system equation (19) and (18) have $2(N+3)$ unknown i.e. $\delta_{-1}, \delta_0, \delta_1, \dots, \delta_{N+1}$ and $\sigma_{-1}, \sigma_0, \sigma_1, \dots, \sigma_{N+1}$ and $(N+1) \times 2$ equations which constitutes an over determined system of $2(N+3) \times 2(N+1)$; to make the system solvable, we

reduce the unknowns by imposing boundary conditions such that we could deal with square matrix system dimension. Here, we eliminate boundary terms $\delta_{-1}, \delta_{N+1}$ and $\sigma_{-1}, \sigma_{N+1}$. This leaves us with the require system $(2N + 2) \times (2N + 2)$ matrix which is a bi-triagonal that could be solved by modified Thomas Algorithm.

Stability of the Scheme

We use Von Neumann stability algorithm to check the stability of the scheme. First, we linearize the nonlinear terms uv_x and $(uv)_x$ by considering both u and v as local constants μ_1 and μ_2 respectively. We substitute into (13b) and (14) and obtain the following equation with the variable δ_m :

$$\mu_1 = (\delta_{m-1} + 4\delta_m + \delta_{m+1}) - \frac{3\Delta t}{h^2}(\delta_{m-1} - 2\delta_m + \delta_{m+1}) + \left(\frac{3\alpha\Delta t}{2h}\mu_2 + \frac{3\eta\Delta t}{2h}\mu_1\right)(\delta_{m-1} + \delta_{m+1}) + \frac{3\alpha\Delta t}{2h}(\sigma_{m-1} + \sigma_{m+1})\mu_1 \quad (21)$$

and

$$\mu_2 = (\sigma_{m-1} + 4\sigma_m + \sigma_{m+1}) - \frac{3\Delta t}{h^2}(\sigma_{m-1} - 2\sigma_m + \sigma_{m+1}) + \frac{3\Delta t}{2h}(\sigma_{m+1} + \sigma_{m-1})(\eta\mu_2 + \beta\mu_1) + \frac{3\Delta t}{2h}(\beta\mu_2)(\delta_{m-1} + \delta_{m+1}) \quad (22)$$

From (21) we obtain:

$$\begin{aligned} & \left(1 - \frac{3\Delta t}{h^2} + \frac{3\Delta t}{2h}(\alpha\mu_2 + \eta\mu_1)\right) \cdot \delta_{m-1}^{n+1} + \left(4 + \frac{6\Delta t}{h^2}\right) \cdot \delta_m^{n+1} + \\ & + \left(1 - \frac{3\Delta t}{h^2} + \frac{3\Delta t}{2h}(\alpha\mu_2 + \eta\mu_1)\right) \cdot \delta_{m+1}^{n+1} + \frac{3\Delta t}{2h}(\alpha\mu_2) \cdot (\sigma_{m+1}^{n+1} + \sigma_{m-1}^{n+1}) = \left(1 + \frac{3\Delta t}{h^2} - \frac{3\Delta t}{2h}(\alpha\mu_2 + \eta\mu_1)\right) \cdot \delta_{m-1}^n + \left(4 - \frac{6\Delta t}{h^2}\right) \cdot \delta_m^n + \left(1 + \frac{3\Delta t}{h^2} - \frac{3\Delta t}{2h}(\alpha\mu_2 + \eta\mu_1)\right) \cdot \delta_{m+1}^n - \frac{3\Delta t}{2h}(\alpha\mu_2) \cdot (\sigma_{m+1}^n + \sigma_{m-1}^n) \end{aligned} \quad (23)$$

The equation (23) summarizes and translates to:

$$w_1\delta_{m-1}^{n+1} + w_2\delta_m^{n+1} + w_3\delta_{m+1}^{n+1} + (\sigma_{m+1}^{n+1} + \sigma_{m-1}^{n+1})w = w_4\delta_{m-1}^n + w_5\delta_m^n + w_6\delta_{m+1}^n + (\sigma_{m+1}^n + \sigma_{m-1}^n)w \quad (24)$$

Where $w, w_1, \dots, w_6$ represent the coefficients of $(\sigma_{m+1} + \sigma_{m-1}), \delta_{m-1}, \delta_m, \delta_{m+1}$ in equation (23)

Now, by Von Neumann stability scheme, we affirm that the solution of the discrete scheme (10)-(11) approximates the exact solution $u(x, t)$ of the Coupled Viscuous Burgers' equation (1)-(2). The round off error $\epsilon_{i,j}^n$ due to approximation and defined by:

$$\epsilon_{i,j}^n = |u^{exact} - u^{num}|$$

Since the exact solution must satisfy the discretized equation, the error too must satisfy the discretized equation. Here we assume that the numerical solution too must also satisfy the discretized equation, but we admit that this is only possible in machine precision. We may now reformulate (19)-(20) in terms of their error terms, i.e. replacing $u_{i,j}^n$ with $\epsilon_{i,j}^n$. Obviously, the error and numerical solution have the same growth or decay rate with respect to time. For linear differential equations with periodic boundary condition, the spatial variation of error may be expanded in terms of their Fourier series thus:

$$\in(x) = \sum_{m=1}^M A_m e^{ik_m x};$$

We refer to equation (24) and let $\delta_m^n = A\xi^n e^{imh\varphi}$ and $\sigma_m^n = B\xi^n e^{imh\varphi}$; A and B are the Harmonic amplitudes, φ is the mode number and h is the element size, $i = \sqrt{-1}$. Upon doing the above substitution, we get:

$$|X_2 + iY|\xi^{n+1} = |X_1 - iY|\xi^n,$$

$$\text{so, } H = \frac{|X_1 - iY|\xi^n}{|X_2 + iY|\xi^{n+1}} \quad (25)$$

where,

$$X_2 = 1 - 4K_2 \sin^2\left(\frac{\varphi h}{2}\right) + 2R_2 \cos^2\left(\frac{\varphi h}{2}\right),$$

$$X_1 = 1 + 4K_2 \sin^2\left(\frac{\varphi h}{2}\right) + 2R_2 \cos^2\left(\frac{\varphi h}{2}\right), Y = 6dK_1 \sin(\varphi h).$$

$$K_1 = \frac{3}{2h} \Delta t, K_2 = \frac{3}{2h^2} v \Delta t, d \text{ is a local constant.}$$

We find that $|H| < 1$, implying unconditional stability of our scheme by the Von Neumann stability theorem.

Next, we may now return to our scheme to compute the numerical solutions of the coupled viscous Burger' equation.

Discussion of Results

We perform numerical experiments in order to gain insight into the performance of the current scheme. Here we provided L_∞ and L_2 errors; this was obtained through the following formula:

$$L_\infty = \max_m \{|u_m - U_m|\},$$

$$L_2 = \frac{\sqrt{\sum_{m=0}^N |u_m - U_m|^2}}{\sqrt{\sum_{m=0}^N |u_m|^2}}$$

Where u_m is the exact solution and U_m is the numerical solution.

For our experiment, we consider a coupled viscous Burgers' equation (12a) and (13a) with $\alpha = \beta = 1$ and $\eta = -2$; this reduce equations (12a) and (13a) to the following:

$u_t - u_{xx} - 2uu_x + (uv)_x = 0, v_t - v_{xx} - 2vv_x + (uv)_x = 0$ with the initial condition given by:

$$u(x, 0) = v(x, 0)$$

$= \sin(x)$, boundary condition is sourced from the exact solution $u(x, t)$.

We adopt the exact solution of coupled viscous Burgers' equation of Kaya, (2001) given by:

$$u(x, t) = v(x, t) = e^{-t} \sin(x).$$

The numerical solutions for this has been obtained by considering the domain $x \in [-\pi, \pi]$ with $\Delta t = \frac{1}{1000}$. The solution given in table 1 represent the number of partitions at different time steps, also represented in figure 1 which is the comparison between the numerical

solutions and exact solutions at different time. Because of symmetry in the initial and boundary conditions, results are presented only for $u(x,t)$. The order of convergence is calculated through the formula

$$R = \frac{\log (\text{Error}(N_1)/\text{Error}(N_2))}{\log (N_2/N_1)}$$

Let $h = \frac{1}{200} = 0.005$, time step $\delta_{m+1} = \sigma_{m+1} = t_3 = 1$, $\delta_m = \sigma_m = t_2 = 0.5$, $\delta_{m-1} = \sigma_{m-1} = t_1 = 0.1$, $\Delta t = \frac{1}{1000} = 0.001$, $\alpha = \beta = 1$, $\eta = -2$

$$a_1 = 1 + \frac{\Delta t}{2} (\eta u_x^n + v_x^n)$$

$$a_1 = 1 + \frac{\Delta t}{2} \left[\eta \left(\frac{3}{h} (\delta_{m+1} + \delta_{m-1}) \right) + \frac{3}{h} (\sigma_{m+1} + \sigma_{m-1}) \right]$$

Substituting values we get

$$a_1 = 1 + \frac{0.001}{2} \left[-2 \left(\frac{3}{0.005} (\delta_{m+1} + \delta_{m-1}) \right) + \frac{3}{0.005} (\sigma_{m+1} + \sigma_{m-1}) \right]$$

$$a_1 = 1 + \frac{0.001}{2} \left[\left(\frac{-6}{0.005} + \frac{3}{0.005} \right) (\delta_{m+1} + \delta_{m-1}) \right]$$

$$a_1 = 1 + 0.0005 \left[\left(-\frac{3}{0.005} \right) (1 + 0.1) \right]$$

$$a_1 = 1 + 0.0005(-600 \times 1.1)$$

$$a_1 = 1 + 0.005 \times -660 = 1 - 0.33 = 0.67$$

For, $a_2 = -3 \frac{\Delta t}{h^2}$, we get

$$a_2 = -3 \frac{0.001}{0.005^2} = \frac{0.003}{2.5 \times 10^{-5}} = 120$$

For, $a_3 = \frac{3\Delta t}{2h} (\alpha v^n + \eta u^n)$

$a_3 = \frac{3\Delta t}{2h} [\alpha(\sigma_{m-1} + 4\sigma_m + \sigma_{m+1}) + \eta(\delta_{m-1} + 4\delta_m + \delta_{m+1})]$ substituting the values we get,

$$a_3 = \frac{3(0.001)}{2(0.005)} [(1-2)(\delta_{m-1} + 4\delta_m + \delta_{m+1})]$$

$$a_3 = \frac{0.003}{0.01} [-(0.1 + 4 \times 0.5 + 1)]$$

$$a_3 = 0.3(-3.1) = -0.93$$

For, $a_4 = \frac{3\Delta t}{2h} (\alpha u^n)$

$$a_4 = \frac{3\Delta t}{2h} \alpha (\delta_{m-1} + 4\delta_m + \delta_{m+1})$$

Substituting values in the equation

$$a_4 = \frac{3(0.001)}{2(0.005)} (0.1 + 4 \times 0.5 + 1)$$

$$a_4 = \frac{0.003}{0.01} (3.1) = 0.3 \times 3.1 = 0.93$$

For, $a_5 = \frac{\Delta t}{2} (\alpha u_x^n)$

$$a_5 = \frac{\Delta t}{2} \alpha \left(\frac{3}{h} (\delta_{m-1} + \delta_{m+1}) \right)$$

Substituting values in the equation

$$a_5 = \frac{0.001}{2} \left(\frac{3}{0.005} (0.1 + 1) \right) = \frac{0.003}{0.01} (1.1) = 0.3(1.1) = 0.33$$

Similarly,

$$\text{For, } a_6 = 1 + \frac{\Delta t}{2} (\eta v_x^n + \beta u_x^n)$$

$$a_6 = 1 + \frac{\Delta t}{2} \left[\eta \left(\frac{3}{h} (\sigma_{m+1} + \sigma_{m-1}) \right) + \beta \frac{3}{h} (\delta_{m+1} + \delta_{m-1}) \right]$$

Substituting values we get

$$a_6 = 1 + \frac{0.001}{2} \left[-2 \left(\frac{3}{0.005} (\sigma_{m+1} + \sigma_{m-1}) \right) + \frac{3}{0.005} (\delta_{m+1} + \delta_{m-1}) \right]$$

$$a_6 = 1 + \frac{0.001}{2} \left[\left(\frac{-6}{0.005} + \frac{3}{0.005} \right) (\sigma_{m+1} + \sigma_{m-1}) \right]$$

$$a_6 = 1 + 0.0005 \left[\left(-\frac{3}{0.005} \right) (1 + 0.1) \right]$$

$$a_6 = 1 + 0.0005(-600 \times 1.1)$$

$$a_6 = 1 + 0.005 \times -660 = 1 - 0.33 = 0.67$$

$$\text{For, } a_7 = -3 \frac{\Delta t}{h^2}$$

$$a_7 = -3 \frac{0.001}{0.005^2} = \frac{0.003}{2.5 \times 10^{-5}} = 120$$

$$\text{For, } a_8 = \frac{3\Delta t}{2h} (\beta u^n + \eta v^n)$$

$$a_8 = \frac{3\Delta t}{2h} [\beta (\delta_{m-1} + 4\delta_m + \delta_{m+1}) + \eta (\sigma_{m-1} + 4\sigma_m + \sigma_{m+1})]$$

Substituting values in the equation

$$a_8 = \frac{3(0.001)}{2(0.005)} [(1 - 2)(\sigma_{m-1} + 4\sigma_m + \sigma_{m+1})]$$

$$a_8 = \frac{0.003}{0.01} [-(0.1 + 4 \times 0.5 + 1)]$$

$$a_8 = 0.3(-3.1) = -0.93$$

$$\text{For, } a_9 = \frac{3\Delta t}{2h} (\beta v^n)$$

$$a_9 = \frac{3\Delta t}{2h} \beta (\sigma_{m-1} + 4\sigma_m + \sigma_{m+1})$$

Substituting values in the equation

$$a_9 = \frac{3(0.001)}{2(0.005)} (0.1 + 4 \times 0.5 + 1)$$

$$a_9 = \frac{0.003}{0.01} (3.1) = 0.3 \times 3.1 = 0.93$$

$$\text{For, } a_{10} = \frac{\Delta t}{2} (\beta v_x^n)$$

$$a_{10} = \frac{\Delta t}{2} \beta \left(\frac{3}{h} (\sigma_{m-1} + \sigma_{m+1}) \right)$$

Substituting values in the equation

$$a_{10} = \frac{0.001}{2} \left(\frac{3}{0.005} (0.1 + 1) \right) = \frac{0.003}{0.01} (1.1) = 0.3(1.1) = 0.33$$

Now substituting the value $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}$ and each time step into equation

$$\begin{aligned} & a_1(\delta_{m-1} + 4\delta_m + \delta_{m+1}) + a_2(\delta_{m-1} - 2\delta_m + \delta_{m+1}) + a_3(\delta_{m-1} + \delta_{m+1}) \\ & + a_4(\sigma_{m-1} + \sigma_{m+1}) + a_5(\sigma_{m-1} + \sigma_m + \sigma_{m+1}) = u^n + \frac{\Delta t}{2} u_{xx}^n \end{aligned}$$

and

$$a_6(\sigma_{m-1} + 4\sigma_m + \sigma_{m+1}) + a_7(\sigma_{m-1} - 2\sigma_m + \sigma_{m+1}) + a_8(\sigma_{m-1} + \sigma_{m+1}) \\ + a_9(\delta_{m-1} + \delta_{m+1}) + a_{10}(\delta_{m-1} + \delta_m + \delta_{m+1}) = v^n + \frac{\Delta t}{2} v_{xx}^n$$

We will get the numerical solution using wolfram Alpha,

For Exact solution u_m we take $x \in [-\pi, \pi]$ at each time step, we substitute into $U(x, t) = V(x, t) = e^{-t} \sin x$

Table 1: Maximum Error and Computed Order of Convergence for $U(t, x)$ at different time level

N	L_∞	Ratio	Order of Convergence
32	2.9104E-04		
64	7.2704E-05	4.0030	2.001
125	1.8178E-05	3.9196	1.999
256	4.5497E-05	1.9953	1.998
572	1.1406E-6	3.9806	1.995

Table 2: Relative Errors and Maximum errors of the problem $U(x, t)$ with $\Delta t = 0.001$

Time	L_2 (N=200)	L_∞
0.1	8.21E-06	7.45E-06
0.5	2.49E-05	4.10E-05
1.0	3.00E-05	8.21E-05

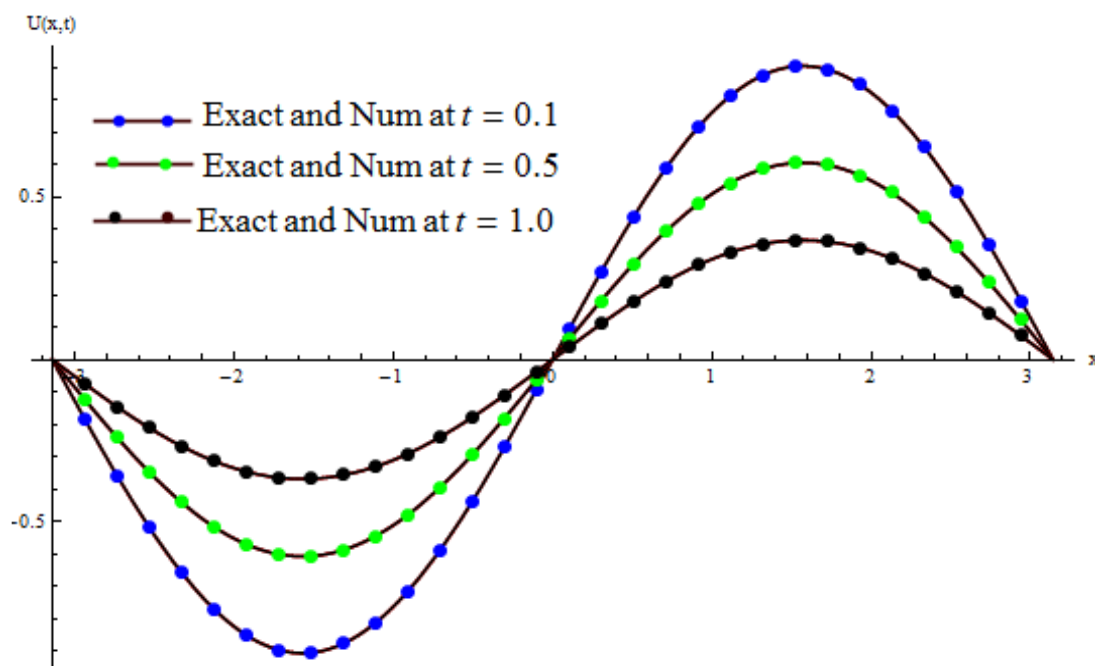


Figure 1: A comparison between the numerical solutions and exact solutions of $U(x, t)$

Conclusion

In this research, Cubic B-Spline plane method was used to solve numerically the one-dimensional Burger's Equation with initial condition and boundary conditions. The cubic trigonometric B-spline was used for interpolating the solutions at each time and using the Von-Neumann method to check the stability. The obtained numerical result showed that the method was efficient, robust and reliable for solving Burgers' Equation accurately even involving high Reynolds numbers for which the exact solutions have failed. The numerical results shown in the Tables (1 and 2) indicate the reliability of results obtained. The obtained solution to the coupled viscous Burgers' equation for various time levels has been compared with the exact solution and we have found that cubic trigonometric B-spline collocation approach has provided more accurate results.

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