

Fréchet Weibull Distribution: Properties and Application

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Abstract

Recent advancements in distributional theory have increasingly focused on constructing flexible probability models capable of capturing complex real-world phenomena more effectively. Addressing this methodological need, this study introduces the Fréchet Weibull distribution, a novel extension of the classical Weibull model generated via the Fréchet-Generalized (Fr-G) family. This compound distribution exhibits enhanced skewness and structural versatility. We systematically derive its fundamental mathematical and statistical properties, alongside parameter estimation via the maximum likelihood method. The practical utility of the model is evaluated using a benchmark real-world dataset—specifically, the 10 mm gauge length data, and benchmarked against established counterparts, including the Lomax-Weibull, Weibull-Weibull, and Alpha Power Weibull distributions. Goodness-of-fit metrics, including the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Hannan-Quinn Information Criterion (HQIC), demonstrate that the proposed model yields superior flexibility and empirical performance compared to existing alternatives.

Keywords: Fréchet-G Family, Weibull Distribution, Maximum Likelihood Estimation, Goodness-of-Fit, Real-Life Data.

Introduction

The Weibull distribution is a continuous probability distribution widely utilized in survival analysis, human mortality modeling, actuarial tables, and growth models. Recently, its application has expanded across numerous scientific and professional fields, including actuarial science, biology, demography, gerontology, and computer science. To enhance its capability, researchers have developed various generalized families of probability distributions. These frameworks introduce additional parameters to existing baseline models, significantly increasing their flexibility for capturing heavily skewed datasets.

A summary of notable generalized families includes the Odd Chen-G family by Anzagra et al. (2022), a new sine family of generalized distributions by Benchiha et al. (2023), Exponentiated-G (EG) by Hassan and Elgarhy (2016), Weibull-G by Bourguignon et al. (2014), the Lomax-G family by Cordeiro et al. (2014), a new generalized Weibull-G family by Cordeiro et al. (2015), a new Weibull-G family by Tahir et al. (2016), and, most recently, the New Fréchet-G Family by Iren et al. (2024).

Building upon these developments, recent literature strongly emphasizes that extended and compound probability distributions consistently outperform classical baseline models in empirical data-fitting applications (Anzagra et al., 2022; Benchiha et al., 2023; Bourguignon et al., 2014; Cordeiro et al., 2019; Umar et al., 2021). Traditional distributions frequently lack the flexibility needed to capture complex phenomena characterized by heavy tails, skewness, and non-monotonic hazard rates. To address these limitations, researchers utilize compounding mechanisms, generator families, and power transformations—such as those explored by Haq and Elgarhy (2018), Ieren and Balogun (2021), and Mohammad (2024)—to inject additional shape and scale parameters into baseline laws. Supported by comprehensive theoretical derivations and empirical validations across diverse fields (Fayomi et al., 2023; Gabanakgosi and Oluyede, 2023; Bhat et al., 2023), these extended frameworks yield superior goodness-of-fit metrics, lower error bounds, and enhanced predictive accuracy compared to nested baseline models.

Motivated by these methodological developments, this paper presents a novel generalization of the Weibull distribution by employing the recently developed Fréchet-G family of probability distributions (Ieren et al., 2024). The Fréchet distribution was selected due to its prominence in extreme value theory, where it excels at modeling the maxima of heavy-tailed processes and proving essential for forecasting rare, high-impact events such as extreme floods, financial crashes, and peak material stresses. We expect this framework to produce a more resilient compound model with superior practical utility, especially within survival analysis, human mortality modeling, actuarial tables, and growth modeling.

Statement of the Problem

The quality and reliability of statistical analyses depend heavily on the accuracy of the assumed probability distribution governing the random variables under investigation. While traditional parametric distributions have long been used to model lifetime data, conventional models frequently fail to capture the complex, real-world behaviors exhibited by modern reliability and survival data.

Recent literature demonstrates that compound probability distributions and extended classical models provide superior goodness-of-fit and flexibility compared to traditional distributions. However, despite these advancements, there remains a need for specialized compound models, such as the Fréchet-Weibull distribution (FrWeiD), capable of effectively modeling complex phenomena, particularly within reliability analysis. Without the development and exploration of such advanced compound distributions, analysts risk utilizing sub-optimal models that reduce the sensitivity, efficiency, and reliability of statistical inferences drawn from lifetime data.

Research Objectives

- i. To formally introduce and define the Fréchet-Weibull distribution.

- ii. To derive key statistical properties of the proposed distribution, including moments, the moment-generating function, the characteristic function, the survival function, the hazard rate function, the quantile function, and entropy.
- iii. To evaluate the empirical performance of the new distribution in comparison with existing probability distributions using real-world data.

The probability density function (pdf) and cumulative distribution function (cdf) of the Weibull distribution are defined respectively as;

$$g(x) = \alpha \beta x^{\beta-1} e^{-\alpha x^\beta} \quad (1.1)$$

and

$$G(x) = 1 - e^{-\alpha x^\beta} \quad (1.2)$$

where $x > 0, \alpha > 0, \beta > 0$, α and β are the scale and shape parameters respectively.

The remainder of this paper is structured as follows: Section 2 introduces the proposed distribution alongside a comprehensive graphical analysis. Section 3 derives its fundamental mathematical and statistical properties. Section 4 details the parameter estimation procedure via maximum likelihood estimation. Finally, Section 5 presents the empirical data analysis and discussion of results, followed by concluding remarks in Section 6.

Formulation of the Fréchet Weibull Distribution (FrWeid)

Definition

According to Ieren et al. (2024), let $g(x; \eta)$ and $G(x; \eta)$ be the PDF and CDF of a random variable X respectively, and η be a $p \times 1$ vector of parameters, if X follows a Fréchet distribution with parameters a and b , that is, $X \sim Fr(a, b)$, then the CDF and PDF of the Fréchet-G family of distributions are defined as:

$$F(x, a, b, \eta) = \exp \left\{ -b^a \left(-\log[1 - G(x; \eta)] \right)^{-a} \right\} \quad (2.1)$$

and

$$f(x; a, b, \eta) = \frac{ab^a g(x; \eta) \left(-\log[1 - G(x; \eta)] \right)^{-a-1}}{[1 - G(x; \eta)]} \exp \left\{ -b^a \left(-\log[1 - G(x; \eta)] \right)^{-a} \right\}$$

(2.2)

respectively, for $a > 0$, $b > 0$, and $x > 0$

where a and b are the two new shape parameters that will enhance the flexibility and shape of the new distribution and η be a $p \times 1$ vector of parameters for the baseline $G(x; \eta)$.

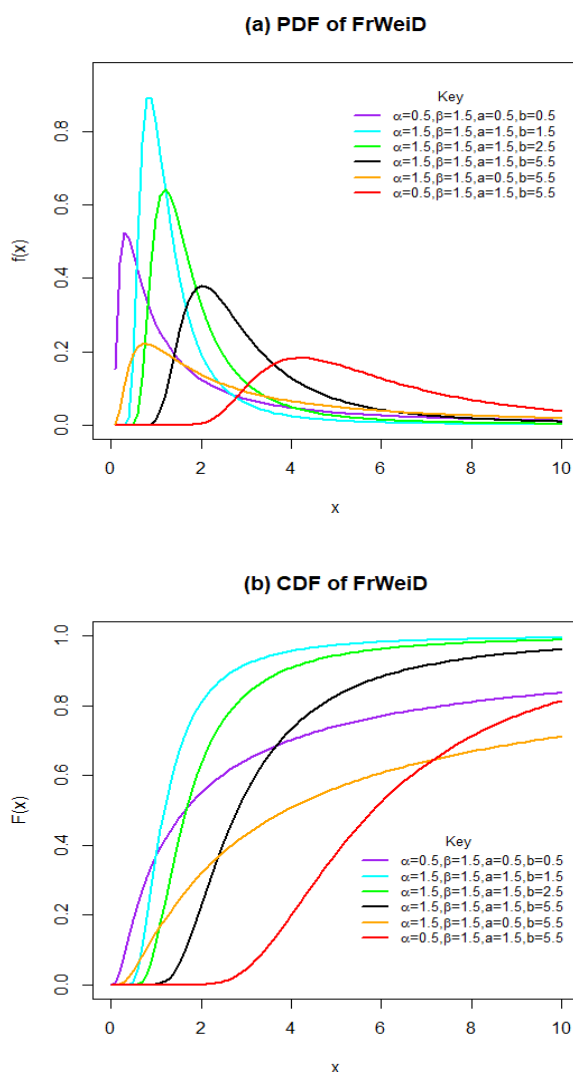
Consequently, the survival function (SF) and hazard function (HF) of the Frechet-Weibull distribution (FrWeid) are obtained from its CDF and PDF as presented in Equation (2.3) and (2.4) respectively below:

$$S(x) = 1 - F(x) = 1 - \exp \left\{ - \left(\frac{b}{\alpha} \right)^a x^{-a\beta} \right\} \quad (2.3)$$

and

$$h(x) = \frac{f(x)}{S(x)} = a\beta \left(\frac{b}{\alpha} \right)^a x^{-a\beta-1} \exp \left\{ - \left(\frac{b}{\alpha} \right)^a x^{-a\beta} \right\} \left(1 - \exp \left\{ - \left(\frac{b}{\alpha} \right)^a x^{-a\beta} \right\} \right)^{-1} \quad (2.4)$$

The Figures 2.1(a)-2.1(d) present the PDF, CDF, survival function (SF), and hazard function (HF) of the FrWeid using arbitrary parameter values.



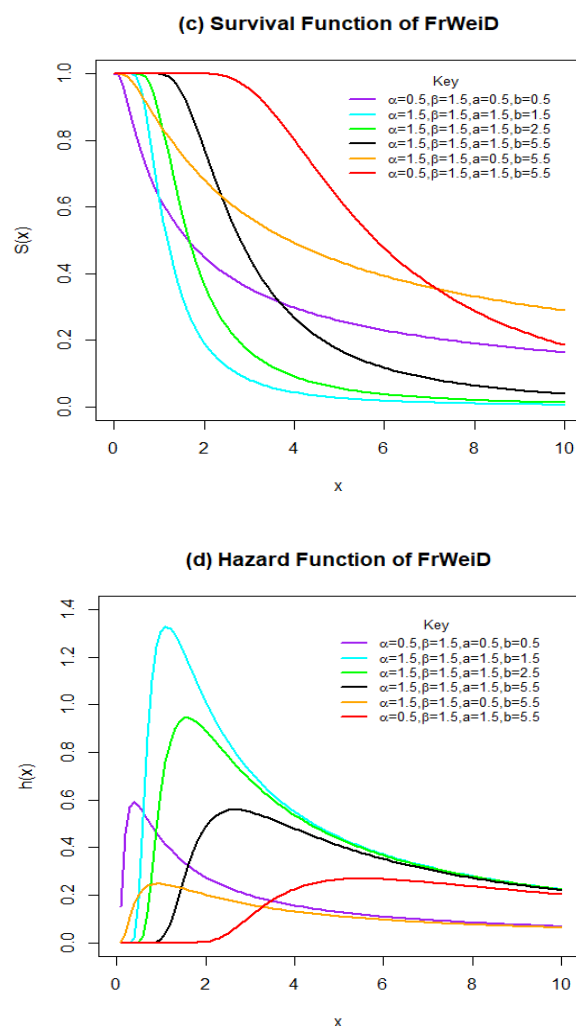


Figure 1: PDF, CDF, SF and HF of FrWeiD for selected parameter values.

The plots of the PDF and HF of the FrWeiD show that the proposed distribution is skewed and flexible and that its shape depends on the values of the parameters. Also, the shapes can be classified as decreasing, unimodal, right-skewed, reversed-J, nearly symmetrical, and asymmetrical shapes, as shown in Figure 1. The plot of CDF and SF all converges to one as expected which also confirms the validity of the FrWeiD.

Mathematical and Statistical Properties of FrWeiD.

Moments

Let X denote a continuous random variable the n^{th} moment of X is given by;

$$\mu'_n = E(X^n) = \int_0^{\infty} x^n f(x) dx \quad (3.1.1)$$

where $f(x)$, the PDF of the FrWeiD is as given below

$$f(x) = a\beta \left(\frac{b}{\alpha}\right)^a x^{-a\beta-1} \exp\left\{-\left(\frac{b}{\alpha}\right)^a x^{-a\beta}\right\} \quad (3.1.2)$$

Substituting Equation (8) in (7) gives;

$$\mu'_n = E(X^n) = \int_0^\infty x^n f(x) dx = \int_0^\infty x^n a\beta \left(\frac{b}{\alpha}\right)^a x^{-a\beta-1} \exp\left\{-\left(\frac{b}{\alpha}\right)^a x^{-a\beta}\right\} dx$$

(3.1.3)

After substitutions and simplification,

$$\mu'_n = E(X^n) = -\left(\frac{\alpha}{b}\right)^{-\frac{n}{\beta}} \int_0^\infty u^{1-\frac{n}{a\beta}-1} e^{-u} du \quad (3.1.4)$$

$$\text{Recall that } \int_0^\infty t^{k-1} e^{-t} dt = \Gamma(k) \quad \text{and that } \int_0^\infty t^k e^{-t} dt = \int_0^\infty t^{k+1-1} e^{-t} dt = \Gamma(k+1)$$

Using the statement above, the n^{th} ordinary moment of X for the FrWeid is obtained as:

$$\mu'_n = E(X^n) = -\left(\frac{\alpha}{b}\right)^{-\frac{n}{\beta}} \Gamma\left(1 - \frac{n}{a\beta}\right) \quad (3.1.5)$$

Moment Generating Function

The moment generating function of a continuous random variable X can be obtained as

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^\infty e^{tx} f(x) dx \quad (3.2.1)$$

Recall that by power series expansion,

$$e^{tx} = \sum_{n=0}^\infty \frac{(tx)^n}{n!} = \sum_{n=0}^\infty \frac{(t)^n}{n!} x^n \quad (3.2.2)$$

Therefore, the moment generating function can also be expressed as:

$$M_x(t) = \sum_{n=0}^\infty \frac{t^n}{n!} \int_0^\infty x^n f(x) dx = \sum_{n=0}^\infty \frac{t^n}{n!} E(X^n) \quad (3.2.3)$$

Using the result in Equation (4.15) and simplifying the integral in (4.13) therefore yields:

$$M_x(t) = \sum_{n=0}^\infty \frac{t^n}{n!} \left[-\left(\frac{\alpha}{b}\right)^{-\frac{n}{\beta}} \Gamma\left(1 - \frac{n}{a\beta}\right) \right] \quad (3.2.4)$$

Characteristics Function

A representation for the characteristics function of a continuous random variable is given by

$$\phi_x(t) = E(e^{itx}) = \int_0^{\infty} e^{itx} f(x) dx \quad (3.3.1)$$

Recall that by power series expansion,

$$e^{itx} = \sum_{n=0}^{\infty} \frac{(itx)^n}{n!} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} x^n \quad (3.3.2)$$

Hence, simple algebra and use of Equation (4.18) produces the following results:

$$\begin{aligned} \phi_x(t) &= E(e^{itx}) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \int_0^{\infty} x^n f(x) dx = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} E(X^n) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} [\mu_n'] \\ \phi_x(t) &= E(e^{itx}) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \left[-\left(\frac{\alpha}{b}\right)^{-\frac{n}{\beta}} \Gamma\left(1 - \frac{n}{a\beta}\right) \right] \end{aligned} \quad (3.3.3)$$

Quantile Function and Median

According to Hyndman and Fan (1996), the quantile function for any distribution is defined

in the form $Q(u) = F^{-1}(u)$ where $Q(u)$ is the quantile function of $F(x)$ for $0 < u < 1$

To derive the quantile function of the FrWeid, the cdf of the FrWeid is considered and inverted according to the above definition as follows:

$$F(x) = \exp\left\{\left(\frac{b}{\alpha}\right)^a x^{-a\beta}\right\} = u \quad (3.4.1)$$

Simplifying Equation (4.20) above gives:

$$Q(u) = \left[-\left(\frac{\alpha}{b}\right)^a \ln(u) \right]^{-\frac{1}{a\beta}} \quad (3.4.2)$$

Using Equation (3.4.2), the median of X from the FrWeid is simply obtained by setting $u = 0.5$ as follows:

$$Median = \left[-\left(\frac{\alpha}{b}\right)^a \ln\left(\frac{1}{2}\right) \right]^{-\frac{1}{a\beta}} \quad (3.4.3)$$

Similarly, random numbers can be simulated from the FrWeid by setting $Q(u) = X$ as follows:

$$X = \left[-\left(\frac{\alpha}{b}\right)^a \ln(u) \right]^{-\frac{1}{a\beta}} \quad (3.4.4)$$

Also, Kenney, J. F., and Keeping, E. S. (1962) defined the Bowley's measure of skewness based on quartiles as:

$$SK = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q\left(\frac{1}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)} \quad (3.4.5)$$

Relatedly, Moors (1988) presented the Moors' kurtosis based on octiles as:

$$KT = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)} \quad (3.4.6)$$

“where $Q(\cdot)$ is calculated by using the quantile function from Equation (20).

Entropy Measurement

The Rènyi entropy (RE) of a random variable X which represents a variation of the uncertainty is defined as:

$$R_{Entr} = \frac{1}{1-\delta} \log \int_{-\infty}^{\infty} f^{\delta}(x) dx \quad (3.5.1)$$

for $\delta > 0$ and $\delta \neq 1$.

$$f(x) = a\beta \left(\frac{b}{\alpha}\right)^a x^{-a\beta-1} \exp\left\{-\left(\frac{b}{\alpha}\right)^a x^{-a\beta}\right\}$$

Now, recall that is the PDF of the FrWeid and using it in Equation (3.5.1) gives:

$$RE = \frac{1}{1-\delta} \log \left[\int_0^{\infty} a^{\delta} \beta^{\delta} \left(\frac{b}{\alpha}\right)^{a\delta} x^{-\delta(a\beta+1)} \exp\left\{-\delta \left(\frac{b}{\alpha}\right)^a x^{-a\beta}\right\} dx \right] \quad (3.5.2)$$

Therefore, solving the Equation (3.5.2) above and simplifying the result gives the Rènyi entropy of a random variate, X following FrWeid is given as:

$$RE = \frac{1}{1-\delta} \log \left[(a\beta)^{\delta-1} \left(\frac{\alpha}{b}\right)^{\frac{\delta-1}{\beta}} \Gamma\left(\delta + \frac{\delta-1}{a\beta}\right) \right] \quad (3.5.3)$$

Maximum Likelihood Estimation of the unknown Parameters of the FrWeid

Let $X_1, X_2, \dots, X_n$ be a sample of size 'n' independently and identically distributed random variables from the FrWeid with unknown parameters, a , b , α , and β defined previously.

The likelihood function of the FrWeid using the pdf in Equation (2.2) is given by:

$$L(\underline{X} | a, b, \alpha, \beta) = \left(a\beta \left(\frac{b}{\alpha}\right)^a \right)^n \prod_{i=1}^n (x_i^{-a\beta-1}) \exp\left\{-\left(\frac{b}{\alpha}\right)^a \sum_{i=1}^n x_i^{-a\beta}\right\}$$

(4.1)

Let the natural logarithm of the likelihood function be, $l = \log L(\underline{X} | a, b, \alpha, \beta)$, therefore, taking the natural logarithm of the function in Equation (4.1) above gives:

$$l = n \log a + n \log \beta + an \log b - an \log \alpha - (a\beta + 1) \sum_{i=1}^n \log x_i - \left(\frac{b}{\alpha}\right)^a \sum_{i=1}^n x_i^{-a\beta} \quad (4.2)$$

Differentiating l partially with respect to a , b , α , and β respectively gives the following results:

$$\frac{\partial l}{\partial a} = \frac{n}{a} + n \log b - n \log \alpha - \beta \sum_{i=1}^n \log x_i - \left(\frac{b}{\alpha}\right)^a \sum_{i=1}^n x_i^{-a\beta} \left[\ln \left(\frac{b}{\alpha}\right) - \beta \ln x_i \right] \quad (4.3)$$

$$\frac{\partial l}{\partial b} = \frac{an}{b} - ab^{a-1} \alpha^{-a} \sum_{i=1}^n x_i^{-a\beta} \quad (4.4)$$

$$\frac{\partial l}{\partial \alpha} = -\frac{an}{\alpha} + ab^a \alpha^{-a-1} \sum_{i=1}^n x_i^{-a\beta} \quad (4.5)$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - a \sum_{i=1}^n \log x_i + a \left(\frac{b}{\alpha}\right)^a \sum_{i=1}^n x_i^{-a\beta} \ln x_i \quad (4.6)$$

Making Equations (4.3), (4.4), (4.5) and (4.6) equal to zero (0) and solving for the solution of the non-linear system of equations produce the maximum likelihood estimates of parameters a , b , α , and β . Note that it is difficult to solve the above equations analytically and therefore the Newton-Raphson's iteration method is applied using computer applications such as Python, R, Matlab or any other suitable software.

Application of the FrWeiD to Real-Life Dataset

This section evaluates the empirical performance of the proposed Fréchet-Weibull distribution (FrWeiD) through an application to a real-world dataset. To benchmark its efficacy, the FrWeiD is compared against several established models, namely the Lomax-Weibull (LWeiD; Cordeiro, M. G. et al. 2014), Weibull-Weibull (WWeiD; Bourguignon., 2014), Alpha Power Weibull (APWeiD; Nassar et al., 2016), Kumaraswamy-Weibull (KWeiD; Cordeiro, M. G. et al. 2010), and the baseline Weibull distribution (WeiD).

Summary descriptive statistics for the dataset alongside the empirical findings are detailed in the tables below. Model selection and goodness-of-fit are rigorously assessed using standard information criteria and empirical distribution function statistics, including the Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), Anderson-Darling (A*), and Cramér-von Mises (W*) tests. For a comprehensive discussion

of these goodness-of-fit measures, see Chen and Balakrishnan (1995). Ultimately, smaller values across these criteria indicate a superior fit to the data

Real-Life Dataset on the Gauge Length of 10mm

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020. (Kundu and Raqab, 2009). Its summary is given as follows:

Table 1: Descriptive Statistics for the dataset

N	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
63	1.901	2.554	2.996	3.421	3.059	5.020	0.38554	0.63285	0.28634

Source: R- Package

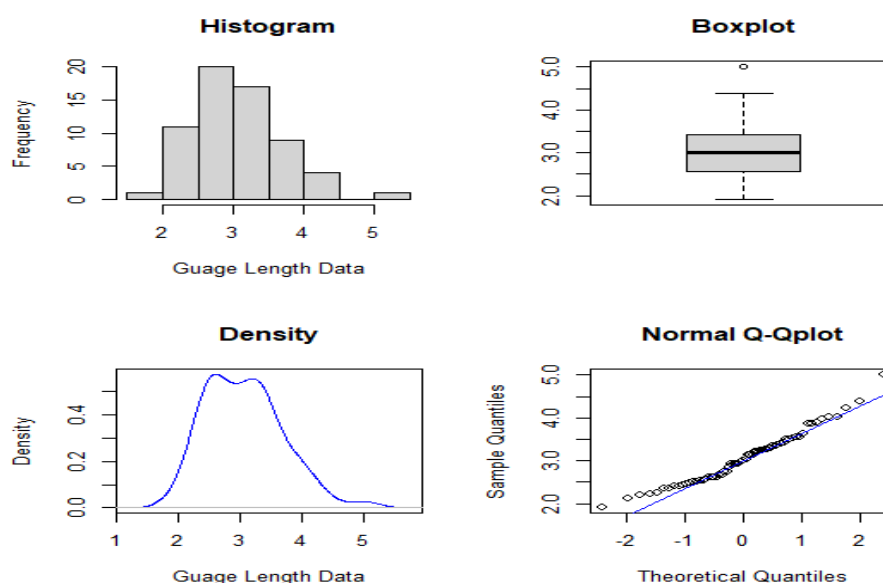
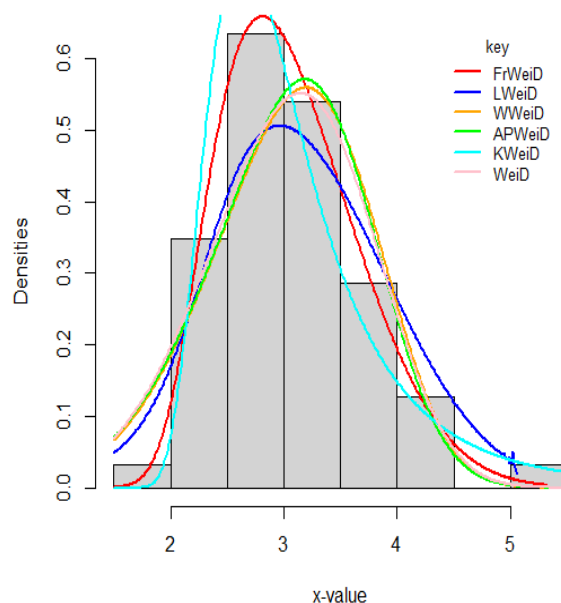


Figure 2: A graphical summary of the Gauge Length dataset. Source; R- Package

As evidenced by the descriptive statistics in Table 2, the empirical dataset exhibits positive (right) skewness, confirming its appropriateness for testing the proposed distribution. To visually appraise the goodness-of-fit, the estimated probability density functions (PDFs) and cumulative distribution functions (CDFs) for the gauge length dataset are illustrated in the figure below.

Estimated Pdfs based on the Guage Length Dataset



Estimated Cdfs based on the Guage Length Dataset

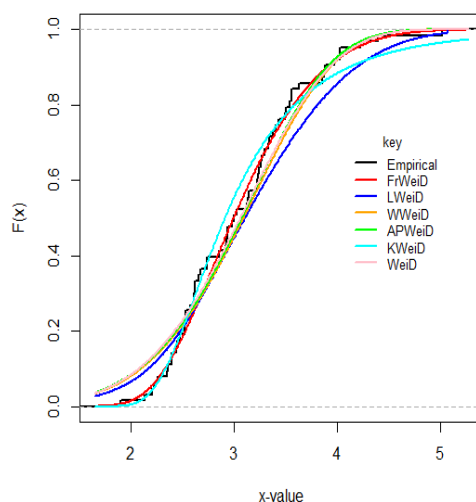


Figure 3: Plots of the estimated densities and CDFs of the fitted distributions based on the Guage length dataset.

Source: R- Package

Table 2: Maximum Likelihood Estimates based on the dataset

Distribution	Parameter Estimates			
FrWeiD	$\hat{a} = 0.3771633481$	$\hat{b} = 2.153708045$	$\hat{\alpha} = 2.52297940$	$\hat{\beta} = 3.25803353$
LWeiD	$\hat{a} = 0.1502521874$	$\hat{b} = 3.402270542$	$\hat{\alpha} = 0.12887509$	$\hat{\beta} = 5.56360087$

Distribution	Parameter Estimates			
WWeiD	$\hat{a} = 0.6068649087$	$\hat{b} = 0.283372106$	$\hat{\alpha} = 0.02971593$	$\hat{\beta} = 11.8047382$
APWeiD	$\hat{a} = 0.0057978005$	$\hat{b} = 4.500062934$	$\hat{\alpha} = 2.19171023$	-
KWeiD	$\hat{a} = 0.1625363665$	$\hat{b} = 3.252880961$	$\hat{\alpha} = 11.0852767$	$\hat{\beta} = 0.20540711$
WeiD	$\hat{a} = 0.0031024230$	$\hat{b} = 4.836437499$	-	-

Source: R- Package

Table 3: The statistics A^* , W^* , AIC, CAIC, BIC and HQIC based on the Guage length data

Distribution	A^*	W^*	AIC	CAIC	BIC	HQIC
FrWeiD	0.2586136	0.04489478	120.6668	119.9771	128.5496	123.3487
KWeiD	0.6420031	0.11496862	126.4940	125.8043	134.3768	129.1759
LWeiD	0.4297238	0.06447436	130.8717	130.1821	138.7546	133.5537
WWeiD	0.9465931	0.13668190	133.8852	133.1955	141.7681	136.5672
APWeiD	0.9479377	0.13753090	131.3432	130.9364	137.3658	133.4651
WeiD	0.8508526	0.12223610	128.3359	128.1359	132.4222	129.8217

Source: R- Package

Discussion of Results and Findings

The graphical analyses of the probability density functions (PDFs) and hazard functions (HFs) demonstrate that the FrWeiD is a highly flexible and skewed distribution whose shape is governed by its parameter values. Specifically, the model accommodates a wide array of distributional forms, including decreasing, unimodal, right-skewed, reversed-J, nearly symmetrical, and asymmetric shapes.

The behavior of the survival function plots further highlights the practical utility of the FrWeiD, indicating its suitability for modeling time- or age-dependent events characterized by decreasing survival rates over time. Additionally, the cumulative distribution function (CDF) and survival function plots confirm that the FrWeiD satisfies the theoretical requirements of a valid probability distribution.

The quantitative findings corroborate these visual insights. Tables 2 and 3 summarize the parameter estimates and goodness-of-fit metrics, namely A^* , W^* , AIC, CAIC, BIC, and HQIC, for the gauge length dataset across six competing models. As detailed in Table 3, the proposed FrWeiD consistently achieves the lowest values across all considered goodness-of-fit criteria when compared to the benchmark models: the Lomax-Weibull distribution (LWeiD; Cordeiro et al., 2014), Weibull-Weibull distribution (WWeiD; Ieren et al., 2019), Alpha Power Weibull distribution (APWeiD; Nassar et al., 2016), Kumaraswamy-Weibull distribution (KWeiD; Cordeiro et al., 2010), and the conventional Weibull distribution (WeiD).

Ultimately, these numerical advantages are reinforced by the visual evidence presented in Figure 3 (estimated densities and cumulative distributions), demonstrating the superior fit

and modeling capacity of the FrWeiD over existing alternatives.

Conclusion

This study establishes the FrWeiD as a highly versatile and theoretically sound probability model capable of capturing diverse empirical behaviors, such as decreasing, unimodal, and right-skewed patterns. Because its survival and hazard profiles effectively represent age- or time-dependent processes, the distribution is well-suited for practical survival analysis. Furthermore, empirical evaluation using the gauge length dataset demonstrates that the FrWeiD outperforms established benchmark models, including the LWeiD, WWeiD, APWeiD, KWeiD, and conventional WeiD by achieving superior goodness-of-fit metrics across all evaluated criteria. These quantitative advantages are consistently validated by graphical diagnostics, confirming the proposed distribution's exceptional capability and reliability for modeling complex data.

Competing Interests

Authors have declared that no competing interests exist.

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